A Subdeterminant Inequality for Normal Matrices

Marvin Marcus* and Kenneth Moore
Algebra Institute
University of California at Santa Barbara
Santa Barbara, California 93106

Submitted by Hans Schneider

ABSTRACT

Let A be an $n\times n$ normal matrix over $\mathbb{C}$, and $Q_{m,n}$ be the set of strictly increasing integer sequences of length m chosen from $1,\ldots,n$. For $\alpha,\beta\in Q_{m,n}$ denote by $A[\alpha|\beta]$ the submatrix obtained from A by using rows numbered α and columns numbered β. For $k\in\{0,1,\ldots,m\}$ we write $|\alpha\cap\beta|=k$ if there exists a rearrangement of $1,\ldots,m$, say $i_1,\ldots,i_k,i_{k+1},\ldots,i_m$, such that $\alpha(i_j)=\beta(i_j)$, $i=1,\ldots,k$, and $\{\alpha(i_{k+1}),\ldots,\alpha(i_m)\}\cap\{\beta(i_{k+1}),\ldots,\beta(i_m)\}=\varnothing$. A new bound for $|\det A[\alpha|\beta]|$ is obtained in terms of the eigenvalues of A when $2m=n$ and $|\alpha\cap\beta|=0$.

Let $\mathcal{U}_n$ be the group of $n\times n$ unitary matrices. Define the nonnegative number

$$\rho_k(A)=\max_{U\in\mathcal{U}_n}|\det(U^*AU)[\alpha|\beta]|,$$

where $|\alpha\cap\beta|=k$. It is proved that

$$\rho_k(A)=\max_{U\in\mathcal{U}_n}|\det(U^*AU)[1,\ldots,m|1,\ldots,k,m+1,\ldots,2m-k]|.$$

Let A be semidefinite hermitian. We conjecture that

$$\rho_0(A)\leqslant\rho_1(A)\leqslant\cdots\leqslant\rho_m(A).$$

These inequalities have been tested by machine calculations.

I. INTRODUCTION

Let A be an $n\times n$ normal matrix over $\mathbb{C}$, $m\geqslant 2$, $n=2m$. We obtain a bound for certain off-diagonal $m\times m$ subdeterminants of A in terms of the eigenvalues of A. The main result is preceded by two combinatorial lemmas.

*Research of this author was supported by AFOSR Contract AF F4962078-C-0030.

0024-3795/80/030129+15$01.75

Previous bounds and a conjecture are discussed in the last section. The plausibility of the conjecture is supported by machine calculations that are discussed along with a method of using Lemma 1 to test their validity. We include below a brief survey of the multilinear and combinatorial methods used to prove the theorem.

Let V be an n-dimensional inner-product space over $\mathbb{C}$ with an orthonormal (o.n.) basis $e_1,\dots,e_n$. Let m be a fixed integer, $1 \leqslant m \leqslant n$, and let $Q_{m,n}$ be the set of $\binom{n}{m}$ strictly increasing integer sequences of length m chosen from $1,\dots,n$. The mth Grassmann space over V, $\bigwedge^m V$, inherits an inner product from the induced inner product in $\bigotimes^m V$. A nonzero element $z \in \bigwedge^m V$ is said to be decomposable if $z = u_1 \wedge \cdots \wedge u_m$ for m linearly independent vectors $u_1,\dots,u_m$. The $\binom{n}{m}$ decomposable alternating tensors $e_\omega^\wedge = e_{\omega(1)} \wedge \cdots \wedge e_{\omega(m)}, \omega \in Q_{m,n}$, comprise an o.n. basis of $\bigwedge^m V$. Let M be an $n \times n$ matrix over $\mathbb{C}$. Let $\alpha, \beta \in Q_{m,n}$, and denote by $\det M[\alpha|\beta]$ the determinant of the submatrix of M lying in rows $\alpha(1),\dots,\alpha(m)$ and columns $\beta(1),\dots,\beta(m)$, and by $\det M(\alpha|\beta)$ the subdeterminant obtained by deleting rows numbered α and columns numbered β. Certain necessary and sufficient conditions [3, p. 6] for decomposability in $\bigwedge^m V$ may be stated as follows:

Let $z \in \bigwedge^m V$,

$$z = \sum_{\omega \in Q_{m,n}} p(\omega) e_\omega^\wedge, \qquad p(\omega) \in \mathbb{C}.$$

Then $z = u_1 \wedge \cdots \wedge u_m$ if and only if there exists an $A \in M_{m,n}(\mathbb{C})$ such that for every $\omega \in Q_{m,n}$

$$p(\omega) = \det A[1,\dots,m|\omega]. \tag{1}$$

In what follows we restrict our attention to elements $u_1 \wedge \cdots \wedge u_m$ chosen from the Grassmannian manifold, i.e., the set of those decomposable elements of unit length in $\bigwedge^m V$. We may also assume that the vectors $u_1,\dots,u_m$ in $u_1 \wedge \cdots \wedge u_m$ are o.n. Thus if $p(\omega)$, $\omega \in Q_{m,n}$, are the components of an element $u_1 \wedge \cdots \wedge u_m$ in the Grassmannian manifold, i.e.,

$$u_1 \wedge \cdots \wedge u_m = \sum_{\omega \in Q_{m,n}} p(\omega) e_\omega^\wedge,$$

then $\sum_{\omega \in Q_{m,n}} |p(\omega)|^2 = 1$.

Let $\omega \in Q_{m,n}$, and define $\omega' \in Q_{m,n}$ to be the sequence complementary to ω in $1,\dots,n$. Let Q be any set of $\frac{1}{2}\binom{n}{m}$ elements of $Q_{m,n}$ for which $Q \cup Q' = Q_{m,n}$, where $Q' = \{\omega' | \omega \in Q\}$. The main result in this paper is stated in terms of any such set Q.

THEOREM. *Let $m \geqslant 2$, $n = 2m$, and let A be an $n \times n$ normal matrix over $\mathbb{C}$ with eigenvalues $\lambda_1,\dots,\lambda_n$. Denote by λ_ω the product $\lambda_{\omega(1)}\cdots\lambda_{\omega(m)}$, for $\omega \in Q_{m,n}$. Then*

$$|\det A[\omega|\omega']| \leqslant \begin{cases} \dfrac{1}{4} \displaystyle\sum_{\omega \in Q} |\lambda_\omega + \lambda_{\omega'}| & \text{if} \quad m=2, \\ \dfrac{1}{2(m+1)} \displaystyle\sum_{\omega \in Q} |\lambda_\omega + (-1)^m \lambda_{\omega'}| & \text{if} \quad m>2. \end{cases}$$

For example, if $m=2$, $n=4$, we can take $Q = \{(12),(13),(14)\}$, $Q' = \{(34),(24),(23)\}$, and $|\det A[12|34]| \leqslant \frac{1}{4}\{|\lambda_1\lambda_2 + \lambda_3\lambda_4| + |\lambda_1\lambda_3 + \lambda_2\lambda_4| + |\lambda_1\lambda_4 + \lambda_2\lambda_3|\}$.

REMARK. Note that if Q is the set of sequences in $Q_{m,n}$ satisfying $\omega(1)=1$, then $Q \cup Q' = Q_{m,n}$, and $|Q| = |Q'| = \frac{1}{2}|Q_{m,n}|$.

II. TWO PRELIMINARY LEMMAS

Let $\alpha, \beta \in Q_{m,n}$, $k \in \{0,1,\dots,m\}$. We say α and β intersect in k places, written $|\alpha \cap \beta| = k$, if there exists a rearrangement of $1,\dots,m$, say $i_1,\dots,i_k$, $i_{k+1},\dots,i_m$, such that

$$\alpha(i_1) < \alpha(i_2) < \cdots < \alpha(i_k)$$

and

$$\alpha(i_{k+1}) < \alpha(i_{k+2}) < \cdots < \alpha(i_m),$$

where

$$\alpha(i_j) = \beta(i_j), \qquad j = 1,\dots,k$$

and

$$\{\alpha(i_{k+1}),\dots,\alpha(i_m)\} \cap \{\beta(i_{k+1}),\dots,\beta(i_m)\} = \varnothing.$$

Let $\mathbb{C}^n$ be the space of n-tuples over $\mathbb{C}$, $e_1,\ldots,e_n$ the standard basis. Denote by $\mathcal{U}_n$ the group of $n\times n$ unitary matrices. For any $U\in\mathcal{U}_n$, $\mu,\nu\in Q_{m,n}$ set $p_\mu(\nu)=\det U[\mu|\nu]$.

LEMMA 1. *Let $\alpha,\beta,\mu\in Q_{m,n}$ $(n\geqslant 4)$, and suppose $|\alpha\cap\beta|=k$, where $k\in\{0,1,\ldots,m-2\}$. Then*

$$|p_\mu(\alpha)p_\mu(\beta)|\leqslant\begin{cases}\dfrac{1}{4} & \textit{if}\quad k=m-2,\\[2ex] \dfrac{1}{2(m-k+1)} & \textit{if}\quad k<m-2.\end{cases}\tag{2}$$

Proof. The result is an application of the following theorem recently obtained by Marcus and Filippenko [4]:

Let $A\in M_n(\mathbb{C})$ $(n\geqslant 4)$ be a normal matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$. Suppose $2\leqslant m<n$ and $\alpha,\beta\in Q_{m,n}$ are sequences such that

$$|\alpha\cap\beta|=k\in\{0,1,\ldots,m-2\}.$$

Then

$$|\det A[\alpha|\beta]|\leqslant\begin{cases}\dfrac{1}{4}E_m(|\lambda_1|,\ldots,|\lambda_n|) & \textit{if}\quad k=m-2,\\[2ex] \dfrac{1}{2(m-k+1)}E_m(|\lambda_1|,\ldots,|\lambda_n|) & \textit{if}\quad k<m-2,\end{cases}\tag{3}$$

where $E_m(t_1,\ldots,t_n)=\sum_{\nu\in Q_{m,n}}\prod_{i=1}^m t_{\nu(i)}$ is the m*th elementary symmetric polynomial.*

Let $U\in\mathcal{U}_n$, $\mu\in Q_{m,n}$, and define the $n\times n$ diagonal matrix $A_\mu=[a_{ij}]$ as follows:

$$a_{ij}=\begin{cases}1 & \text{if}\quad i=j=\mu(p),\ 1\leqslant p\leqslant m,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $U^*A_\mu U$ is normal with the same eigenvalues as A_μ. Since $a_{11},\ldots,a_{nn}$ are the eigenvalues of A_μ, and $E_m(|a_{11}|,\ldots,|a_{nn}|)=1$, we have from (3) that

$$|\det(U^*A_\mu U)[\alpha|\beta]|\leqslant\begin{cases}\dfrac{1}{4} & \text{if}\quad k=m-2,\\[2ex] \dfrac{1}{2(m-k+1)} & \text{if}\quad k<m-2.\end{cases}\tag{4}$$

Applying the Cauchy-Binet theorem, it follows that

$$\det(U^*A_\mu U)[\alpha|\beta] = \sum_{\nu,\omega\in Q_{m,n}} \det U^*[\alpha|\nu]\det A_\mu[\nu|\omega]\det U[\omega|\beta]$$

$$= \sum_{\nu\in Q_{m,n}} \overline{\det U[\nu|\alpha]}\,\delta_{\nu,\mu}\det U[\nu|\beta]$$

$$= \overline{p_\mu(\alpha)}\,p_\mu(\beta). \tag{5}$$

Substituting (5) in (4), we obtain (2). ■

LEMMA 2. *Let $u_1,\dots,u_n$ be an o.n. basis of $\mathbb{C}^n$ with respect to the standard inner product. Suppose*

$$u_1\wedge\cdots\wedge u_m = \sum_{\omega\in Q_{m,n}} p(\omega)e_\omega^\wedge$$

and

$$u_{m+1}\wedge\cdots\wedge u_n = \sum_{\omega\in Q_{m,n}} q(\omega)e_\omega^\wedge.$$

Let U be the $n\times n$ unitary matrix whose rows are the vectors $u_1,\dots,u_n$. Then

$$\overline{q(\omega)} = \overline{\det U}\,(-1)^{m(m+1)/2+s(\omega')}p(\omega'),\qquad \omega\in Q_{m,n},$$

where $s(\omega)$ is the sum of the integers in ω.

Proof. For any $n\times n$ matrix M define the mth supplementary compound of M [3, p. 42] by

$$C_m^*(M) = \left[(-1)^{s(\alpha)+s(\beta)}\det M(\alpha|\beta)\right],\qquad \alpha,\beta\in Q_{m,n}.$$

The Laplace expansion theorem becomes

$$C_m^*(M)C_m(M^{\mathrm{T}}) = (\det M)I_{\binom{n}{m}}. \tag{6}$$

Since $C_m(U)$ is unitary [2, p. 119], it follows from (6) that

$$C_m^*(U) = (\det U)\,\overline{C_m(U)},$$

where $\overline{C_m(U)} = \left[\overline{\det U[\alpha|\beta]}\right]$. In terms of matrix entries we have

$$(-1)^{s(\alpha)+s(\beta)}\det U(\alpha|\beta) = (\det U)\,\overline{\det U[\alpha|\beta]}\,.$$

Thus

$$(-1)^{m(m+1)/2+s(\omega)}\det U[m+1,\dots,n|\omega'] = (\det U)\,\overline{\det U[1,\dots,m|\omega]}\,. \quad (7)$$

Now $q(\omega) = \det U[m+1,\dots,n|\omega]$ and $p(\omega) = \det U[1,\dots,m|\omega]$, so from (7) it follows that

$$\overline{q(\omega)} = (-1)^{m(m+1)/2+s(\omega')}\,\overline{(\det U)}\,p(\omega'), \qquad \omega \in Q_{m,n}. \qquad \blacksquare$$

III. PROOF OF THE THEOREM

We first note that for any $\omega \in Q_{m,n}$

$$\det A[\omega|\omega'] = \det PAP^{\mathsf{T}}[1,\dots,m|m+1,\dots,2m]$$

for the permutation matrix $P = [\delta_{i,\varphi(j)}]$ corresponding to the permutation

$$\varphi = \begin{pmatrix} \cdots & \omega(i) & \cdots & \omega'(i) & \cdots \\ \cdots & i & \cdots & m+i & \cdots \end{pmatrix}, \qquad 1 \leqslant i \leqslant m$$

(see Lemma 3 for a more general result). The matrix PAP^{T} is normal and has the same eigenvalues as A. Thus it suffices to obtain the bound for $|\det A[1,\dots,m|m+1,\dots,2m]|$.

Let $u_1,\dots,u_n$ be an o.n. basis of $\mathbb{C}^n$. Let U be the $n\times n$ unitary matrix whose rows are $u_1,\dots,u_n$, and T be the normal transformation on $\mathbb{C}^n$ which has A^{T} as its matrix representation with respect to $u_1,\dots,u_n$. Let $e_1,\dots,e_n$ be an o.n. basis of eigenvectors of T. Then

$$\det A[1,\dots,m|m+1,\dots,n] = (C_m(T)u_1\wedge\cdots\wedge u_m, u_{m+1}\wedge\cdots\wedge u_n).$$

Since $n=2m$, $(-1)^{n(n+1)/2} = (-1)^m$, and $s(\omega) + s(\omega') = n(n+1)/2$, for

$\omega \in Q_{m,n}$. Moreover, from Lemma 1, $|\omega \cap \omega'| = 0$ implies

$$|p(\omega)p(\omega')| \leq \begin{cases} \dfrac{1}{4} & \text{if} \quad m=2, \\ \dfrac{1}{2(m+1)} & \text{if} \quad m>2. \end{cases}$$

Hence

$$\begin{aligned}
&|\det A[1,\ldots,m|m+1,\ldots,n]| \\
&\quad = |(C_m(T)u_1 \wedge \cdots \wedge u_m, u_{m+1} \wedge \cdots \wedge u_n)| \\
&\quad = \left| \left(C_m(T) \sum_{\omega \in Q_{m,n}} p(\omega) e_\omega^\wedge, \sum_{\omega \in Q_{m,n}} q(\omega) e_\omega^\wedge \right) \right| \\
&\quad = \left| \sum_{\omega \in Q_{m,n}} \lambda_\omega p(\omega) \overline{q(\omega)} \right| \\
&\quad = \left| \sum_{\omega \in Q} \lambda_\omega p(\omega) \overline{q(\omega)} + \sum_{\omega' \in Q'} \lambda_{\omega'} p(\omega') \overline{q(\omega')} \right| \\
&\quad = \left| \sum_{\omega \in Q} \overline{\det U} (-1)^{m(m+1)/2 + s(\omega')} \lambda_\omega p(\omega) p(\omega') \right. \\
&\qquad \left. + \sum_{\omega' \in Q'} \overline{\det U} (-1)^{m(m+1)/2 + s(\omega)} \lambda_{\omega'} p(\omega') p(\omega) \right| \\
&\quad = \left| \sum_{\omega \in Q} \overline{\det U} (-1)^{m(m+1)/2 + s(\omega')} \left[\lambda_\omega + (-1)^m \lambda_{\omega'} \right] p(\omega) p(\omega') \right| \\
&\quad \leq \begin{cases} \dfrac{1}{4} \displaystyle\sum_{\omega \in Q} |\lambda_\omega + \lambda_{\omega'}| & \text{if} \quad m=2, \\ \dfrac{1}{2(m+1)} \displaystyle\sum_{\omega \in Q} |\lambda_\omega + (-1)^m \lambda_{\omega'}| & \text{if} \quad m>2 \end{cases}
\end{aligned}$$

by Lemma 1. ■

IV. DISCUSSION OF RESULTS

We record here a lemma to introduce an important normalization.

LEMMA 3. *Let* $\alpha,\beta \in Q_{m,n}$, $2m \leqslant n$, $k \in \{0,1,\ldots,m\}$, *and* $|\alpha \cap \beta| = k$. *Then for any* $n \times n$ *matrix* $A = [a_{ij}]$ *there exist permutations* $\varphi \in S_n$ *and* $\sigma \in S_m$ *with corresponding permutation matrices* $P = [\delta_{i,\varphi(j)}]$ *and* $Q = [\delta_{i,\sigma(j)}]$ *such that*

$$Q\big(PAP^{\mathsf{T}}[1,\ldots,m|1,\ldots,k,m+1,\ldots,2m-k]\big)Q^{\mathsf{T}} = A[\alpha|\beta].$$

Proof. If $k = m$, then $\alpha = \beta$. Thus $A[\alpha|\alpha]$ is principal and we need only set

$$\varphi = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

and

$$\sigma = \begin{pmatrix} \cdots & \alpha(j) & \cdots \\ \cdots & j & \cdots \end{pmatrix}, \qquad j = 1,\ldots,m.$$

If $k < m$, define

$$\varphi = \begin{pmatrix} \cdots & \alpha(i_j) & \cdots & \beta(i_{k+l}) & \cdots \\ \cdots & j & \cdots & m+l & \cdots \end{pmatrix}, \qquad j = 1,\ldots,m, \quad l = 1,\ldots,m-k,$$

where those integers among $1,\ldots,n$ other than $\alpha(i_1),\ldots,\alpha(i_m)$, $\beta(i_{k+1}),\ldots,\beta(i_m)$ are assigned in a one-to-one fashion to $2m-k+1,\ldots,n$. Define

$$\sigma = \begin{pmatrix} \cdots & j & \cdots \\ \cdots & i_j & \cdots \end{pmatrix}, \qquad j = 1,\ldots,m.$$

We compute that

$$\begin{aligned} (PAP^{\mathsf{T}})_{ij} &= \sum_{s,t}^{n} P_{is} A_{st} P_{jt} \\ &= \sum_{s,t}^{n} \delta_{i,\varphi(s)} A_{st} \delta_{j,\varphi(t)} \\ &= A_{\varphi^{-1}(i),\varphi^{-1}(j)}. \end{aligned}$$

Let $B = PAP^{\mathsf{T}}[1,\dots,m|1,\dots,k,m+1,\dots,2m-k]$. Then

$$B_{ij} = \begin{cases} a_{\varphi^{-1}(i),\varphi^{-1}(j)}, & 1 \leqslant j \leqslant k, \\ a_{\varphi^{-1}(i),\varphi^{-1}(m-k+j)}, & k+1 \leqslant j \leqslant m. \end{cases}$$

As above,

$$(QBQ^{\mathsf{T}})_{ij} = B_{\sigma^{-1}(i),\sigma^{-1}(j)}.$$

Since $\sigma^{-1}(i_j)$, $1 \leqslant j \leqslant m$, we have

$$\begin{aligned}(QBQ^{\mathsf{T}})_{i_p i_q} &= B_{\sigma^{-1}(i_p),\sigma^{-1}(i_q)} \\ &= \begin{cases} a_{\varphi^{-1}\sigma^{-1}(i_p),\varphi^{-1}\sigma^{-1}(i_q)}, & 1 \leqslant q \leqslant k, \\ a_{\varphi^{-1}\sigma^{-1}(i_p),\varphi^{-1}(m-k+\sigma^{-1}(i_q))}, & k+1 \leqslant q \leqslant m. \end{cases} \end{aligned} \tag{8}$$

Setting $q = k+l$ for $k+1 \leqslant q \leqslant m$, we see that

$$m-k+q = m+l, \qquad 1 \leqslant l \leqslant m-k.$$

From the structure of φ we conclude, for $1 \leqslant l \leqslant m-k$ and $k+1 \leqslant q \leqslant m$, that

$$\begin{aligned}\varphi^{-1}\big(m-k+\sigma^{-1}(i_q)\big) &= \varphi^{-1}(m+l) \\ &= \beta(i_{k+l}) \\ &= \beta(i_q).\end{aligned}$$

Hence (8) is equal to $a_{\alpha(i_p),\beta(i_q)}$. Since $\{i_1,\dots,i_m\} = \{1,\dots,m\}$, the proof is complete. ∎

The bound obtained in this paper deals with the case $|\alpha \cap \beta| = 0$. A comparison of this bound and the one obtained in (3) is instructive. In (3) we see for $|\alpha \cap \beta| = 0$ that

$$|\det A[\alpha|\beta]| \leqslant \begin{cases} \dfrac{1}{4} \displaystyle\sum_{\omega \in Q_{m,n}} |\lambda_\omega|, & m = 2, \\ \dfrac{1}{2(m+1)} \displaystyle\sum_{\omega \in Q_{m,n}} |\lambda_\omega|, & m > 2. \end{cases}$$

Recalling the bound obtained in the Theorem, we have by the triangle inequality that

$$\sum_{\omega \in Q} |\lambda_\omega + (-1)^m \lambda_{\omega'}| \leqslant \sum_{\omega \in Q_{m,n}} |\lambda_\omega|.$$

Hence the new bound is a refinement of the previous bound.

Let $\alpha, \beta \in Q_{m,n}$, $k \in \{0, 1, \ldots, m\}$, and $|\alpha \cap \beta| = k$. For any normal matrix A define the nonnegative number

$$\rho_k(A) = \max_{U \in \mathfrak{U}_n} |\det U^* A U[\alpha | \beta]|.$$

We see from Lemma 3 that

$$\rho_k(A) = \max_{U \in \mathfrak{U}_n} |\det U^* A U[1, \ldots, m | 1, \ldots, k, m+1, \ldots, 2m-k]|.$$

None of the bounds obtained have been shown to be $\rho_k(A)$, the best bound. For a bound to equal $\rho_k(A)$ it would suffice for the bound to equal $|\det U_0^* A U_0[1, \ldots, m | m+1, \ldots, 2m-k]|$ for some unitary matrix U_0. For example, take $n=4$, $m=2$, $\alpha = (12)$, $\beta = (34)$, and set $A = \operatorname{diag}(1,1,0,0)$. It follows from (3) that $|\det U^* A U[\alpha | \beta]| \leqslant \frac{1}{4}$ for any $U \in \mathfrak{U}_4$. Let

$$U_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Then $|\det U_0^* A U_0[\alpha | \beta]| = \frac{1}{4}$, so that $\rho_0(A) = \frac{1}{4}$. For a real symmetric A it would be interesting to know whether $\rho_k(A)$ is an achievable value of $|\det U^* A U[\alpha | \beta]|$ with a real orthogonal U.

Let A be hermitian with eigenvalues $\lambda_1 \leqslant \cdots \leqslant \lambda_n$. Let $\lambda_{\max} = \max_{\omega \in Q_{m,n}} \prod_{i=1}^m \lambda_{\omega(i)}$ and $\lambda_{\min} = \min_{\omega \in Q_{m,n}} \prod_{i=1}^m \lambda_{\omega(i)}$. The numerical range of A [1, p. 112], i.e., the totality of values (Au, u), $\|u\| = 1$, fills the interval $[\lambda_1, \lambda_n]$. Thus

$$\{(C_m(A) u^\wedge, u^\wedge) : \|u^\wedge\| = 1\} = \{\det U^* A U[\alpha | \alpha] : \alpha \in Q_{m,n},\ U \in \mathfrak{U}_n\}$$
$$\subseteq [\lambda_{\min}, \lambda_{\max}]. \tag{9}$$

Since the eigenvectors of $C_m(A)$ are decomposable, $\lambda_{\max}$ and $\lambda_{\min}$ are achievable values in (9). Thus

$$\rho_m(A)=\max\{|\lambda_{\max}|,|\lambda_{\min}|\}. \tag{10}$$

Let $\alpha,\beta\in Q_{m,n}$, $k\in\{0,1,\dots,m-1\}$, and $|\alpha\cap\beta|=k$. Marcus and Robinson [5] proved that *the set*

$$\{\det U^*AU[\alpha|\beta]:U\in\mathfrak{U}_n\}$$

is a closed disc centered at the origin with radius

$$\rho_k(A)\leqslant\frac{\lambda_{\max}-\lambda_{\min}}{2}. \tag{11}$$

Moreover, if the sequences $\omega,\gamma\in Q_{m,n}$ *satisfying* $\lambda_\omega=\lambda_{\max}$, $\lambda_\gamma=\lambda_{\min}$ *are unique, then*

$$\rho_k(A)=\frac{\lambda_{\max}-\lambda_{\min}}{2}$$

if and only if

$$|\alpha\cap\beta|=m-1$$

and

$$|\omega\cap\gamma|=m-1.$$

In other words, for certain rather typical hermitian A the Marcus-Robinson result implies the inequality

$$\rho_k(A)<\rho_{m-1}(A)=\frac{\lambda_{\max}-\lambda_{\min}}{2}, \tag{12}$$

for $k\in\{0,1,\dots,m-2\}$.

If A is semidefinite it follows from (10) and (11) that

$$\rho_k(A)\leqslant\rho_m(A) \tag{13}$$

for $k \in \{0,1,\dots,m-1\}$. Noting (12) and (13) and the dependence of the coefficient $1/2(m-k+1)$ on k in the Marcus-Filippenko bound (3), one is led to conjecture for semidefinite A that

$$\rho_0(A) \leqslant \rho_1(A) \leqslant \cdots \leqslant \rho_{m-1}(A) \leqslant \rho_m(A).$$

This conjecture is to date unresolved. We have completed some preliminary computer searches to test it.

Clearly it may be assumed that A is the diagonal matrix $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$. Set $\alpha=(1,\dots,m)$ and $\alpha_k=(1,\dots,k,m+1,\dots,2m-k)$. Then $|\alpha \cap \alpha_k| = k$. As in Lemma 1, it follows from the Cauchy-Binet theorem that

$$\det U^*AU[\alpha|\alpha_k] = \sum_{\mu \in Q_{m,n}} \lambda_\mu \overline{p_\mu(\alpha)}\, p_\mu(\alpha_k).$$

This observation suggests that as we range over the unitary matrices generated for the computations, knowledge of the distribution, at least in modulus, of the products

$$p_\mu(\alpha) p_\mu(\alpha_k) \tag{14}$$

would be a useful check on the validity of the computer results. From

TABLE 1

A	n	m	$k=0$	$k=1$	$k=2$	$k=3$	max	min	#
1,1,0,0	4	2	.24331	.39261			x		100
1,1,0,0	4	2	.02975	.00120				x	100
1,0,1,0	4	2	.24533	.43773			x		450
1,0,1,0	4	2	.00045	.00465				x	450
0,1,1,0	4	2	.23949	.41863			x		450
0,1,1,0	4	2	.00244	.00195				x	450
0,0,1,1	4	2	.24749	.43623			x		450
0,0,1,1	4	2	.00257	.00216				x	450
1,1,1,0,0,0	6	3	.07250	.11268	.24178		x		100
1,1,1,0,0,0	6	3	.00147	.00146	.00083			x	100
1,0,1,0,0,1	6	3	.10020	.14431	.18725		x		450
1,0,0,0,1,1	6	3	.08811	.12971	.15186		x		450
1,1,1,1,0,0,0,0	8	4	.02324	.04494	.09652	.19462	x		100
1,1,1,1,0,0,0,0	8	4	.00049	.00092	.00145	.00048		x	100

Lemma 1 we know that

$$0 \leqslant |p_\mu(\alpha)p_\mu(\alpha_k)| \leqslant \begin{cases} \dfrac{1}{4} & \text{if} \quad k=m-2, \\ \dfrac{1}{2(m-k+1)} & \text{if} \quad k<m-2. \end{cases} \tag{15}$$

This upper bound, which we saw to be taken on in the case $n=4$, $m=2$, $\mu=(12)$, suggests a theoretical distribution of the products (14). So by using the A_μ, $\mu \in Q_{m,n}$, defined in the proof of Lemma 1 as test matrices, we can check the distribution of the products (14).

The subdeterminants are obtained using standard FORTRAN subroutines and were run on the AS/6 in the UCSB Computer Center. The unitary matrices are obtained by generating $n(n+1)/2$ "pseudorandom" complex numbers to obtain an $n \times n$ skew-hermitian matrix S. The Cayley transform of S is $(I_n - S)(I_n + S)^{-1}$, a unitary matrix.

The two tables presented here contain the data obtained from the computations. Table 1 lists the moduli of the products (14). An x appears in the column labeled "min" ("max") if the datum listed is the minimum (maximum) value taken on for that run. The last column of Table 1, labeled #, lists the number of unitary matrices generated for that run. Table 2 contains data obtained from hermitian semidefinite matrices. It suffices to consider positive semidefinite matrices, for if $A \leqslant 0$, then $-A \geqslant 0$ and

$$|\det U^*AU[\alpha|\beta]| = |\det\{U^*(-A)U\}[\alpha|\beta]|.$$

The first column of either table includes the eigenvalues of the diagonal matrix A; A is an $n \times n$ matrix and we consider $m \times m$ subdeterminants. Due to the normalization justified by Lemma 3 it suffices to consider the values

$$|\det U^*AU[1,\dots,m|1,\dots,k,m+1,\dots,2m-k]| \tag{16}$$

as U runs over a finite set of "randomly" generated unitary matrices. The columns labeled $k=0$, $k=1$, $k=2$, $k=3$, and $k=4$ in Table 2 contain the maximum values obtained for the numbers (16). In Table 2 one hundred unitary matrices were generated for each run.

TABLE 2

A	n	m	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$1,\frac{1}{2},\frac{1}{3},\frac{1}{4}$	4	2	0.03667	0.11762	0.41957		
$1,2,3,4$	4	2	0.88002	2.3705	8.4458		
$10,100,0,0.001$	4	2	243.29	392.61	769.71		
$1,0,0,\frac{1}{4}$	4	2	0.05748	0.07299	0.11471		
$1,2,3,4,5$	5	2	1.0314	3.6461	12.359		
$1,0,\frac{1}{2},10,5$	5	2	7.6828	11.539	19.721		
$1,2,3,4,5,6$	6	2	1.6436	4.2506	14.266		
$1,2,3,40,5,100$	6	2	620.57	666.20	1098.9		
$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6}$	6	2	0.03345	0.09892	0.38888		
$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6}$	6	3	0.00216	0.00621	0.02610	0.13324	
$1,2,3,40,5,100$	6	3	1098.2	1386.3	1624.5	2959.7	
$1000,100,10,1,0,0.01$	6	3	67,324	105,170	221,070	650,600	
$30,20,10,0,0.01,0.001$	6	3	434.84	675.60	1450.8	3892.9	
$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8}$	8	2	0.02012	0.07197	0.37369		
$1,\frac{1}{8},\frac{1}{2},\frac{1}{7},\frac{1}{3},\frac{1}{6},\frac{1}{4},\frac{1}{5}$	8	2	0.04512	0.08232	0.29116		
$1,2,3,4,5,6,7,8$	8	2	1.9060	6.3162	19.145		
$8,1,2,70,3,6,55,\frac{1}{3}$	8	2	226.97	266.19	378.16		
$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8}$	8	3	0.00203	0.00503	0.02261	0.12924	
$1,\frac{1}{8},\frac{1}{2},\frac{1}{7},\frac{1}{3},\frac{1}{6},\frac{1}{4},\frac{1}{5}$	8	3	0.00220	0.00856	0.01779	0.10749	
$80,10,70,60,50,30,20,40$	8	3	3442.2	12,925	47,310	201,440	
$1,2,3,4,5,6,7,8$	8	3	3.0235	4.2983	15.175	74.860	
$8,1,2,70,3,6,55,\frac{1}{3}$	8	3	944.15	1769.7	3690.2	4351.8	
$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8}$	8	4	0.00011	0.00027	0.00119	0.00634	0.03240
$1,\frac{1}{8},\frac{1}{2},\frac{1}{7},\frac{1}{3},\frac{1}{6},\frac{1}{4},\frac{1}{5}$	8	4	0.00011	0.00033	0.00112	0.00494	0.01757
$80,10,70,60,50,30,20,40$	8	4	51,800	216,955	480,249	2,480,682	10,100,225
$1,2,3,4,5,6,7,8$	8	4	4.4268	5.7196	21.873	89.545	282.17
$2,4,90,0,0,\frac{1}{2},85,\frac{1}{4}$	8	4	1592.4	2512.7	3027.8	8223.4	14,684
$8,1,2,70,3,6,55,\frac{1}{3}$	8	4	2162.9	5253.0	7693.2	9794.0	21,919

REFERENCES

1 P. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, N. J., 1967.

2 M. Marcus, *Finite Dimensional Multilinear Algebra*, Part 1, Pure and Applied Mathematics Series No. 23, Marcel Dekker, New York, 1973.

3 M. Marcus, *Finite Dimensional Multilinear Algebra*, Part 2, Pure and Applied Mathematics Series No. 23, Marcel Dekker, New York, 1975.

4 M. Marcus and Ivan Filippenko, Inequalities connecting eigenvalues and non-principal subdeterminants, in *Proceedings of the 2nd International Conference on General Inequalities at Oberwolfach*, to appear.
5 M. Marcus and Herbert Robinson, Bilinear functionals on the Grassmannian manifold, *Linear and Multilinear Algebra* 3: 215–225 (1975).

Received 3 September 1979